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# Some properties of subspace convex-cyclic operators

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**Abstract**

On a Banach space *X,* a bounded linear operator *A* is a called subspace convex-cyclic associated with *W* as a subspace, if the set $Orb(A,x)\bigcap W$ is dense in *W* for a vector $x\in X$ . In this work, we use Hahn- Banach Theorem to show that the extending linear functional preserve subspace convex-cyclic operator property. Also, the algebraic structures of subspace convex-cyclic vectors can be determined, such as the spectrum.

## 1. Introduction

The study of subspace hypercyclic operator was started first by [10], whenever at some time [9] was published another article on it, after mentioning some properties, even though the general idea can be found in [3] and [7] since H. Rezai [12] and [8] answered some questions that asked in [10] affirmatively for that time the study focused on subspace like *W*. This paper, is a continuation of authors' work [1] on subspace convex-cyclic operators even though in [1] we define subspace convex-cyclic, so in this article, we prove some properties on it. As a consequences work, we clarified that Hahn-Banach Theorem has a great act in **Theorem 2.8** for extending a linear functional and the operator remains subspace convex-cyclic. Also, we show that the norm of an operator has affected been subspace convex-cyclic or not. On the other hand, before section two finished, we add projection to the operator.

In the third section, spectrum with the unit circle $\mathbf{T}$ where $\mathbf{T}=\{z\in\mathbf{C}:|z|=1\}$ has developed our idea for checking while the operator been subspace convex-cyclic, as a result of the spectrum, which guarantees for having dense range.

Finally, the kernel of the subspace convex-cyclic operator leads us to think about the invariant subspace, which can be found in **Theorem 3.7**.

We denote the unit disc by **D,** where $\mathbf{D}=\{z\in\mathbf{C}:|z|<1\}$. Also let *H* be an infinite-dimensional (**R** or **C**) Hilbert space with the separable properties. Meanwhile, we have a subspace *W* of *H*. We assume that *W* to be closed. And the algebra of all bounded linear operators acting on the Hilbert space *H* be denoted by **B**(*H*).

**Definition 1.1 [1]**
Let $A \in \mathbf{B}(H)$ and let $W$ be a nonempty subspace of $H$. We assign $A$ as a **subspace convex-cyclic** operator over $W$ if there exists $x \in H$ such that $Orb(A,x) \bigcap W$ is dense in $W$ such a vector $x$ is said to be a subspace-convex-cyclic vector.
Where

$$Orb(A,x) = \{ \; P(A)x : P \; is \quad convex \quad polynomial \; \},$$

$$= \{ \; P(A)x : P(A) := a_0 + a_1 A + ... a_n A^n, n \in \mathbf{N}, \quad \sum_{i=0}^{n} a_i = 1 \; \}.$$

Note: From now, we will use **Sub-Conv-Cyc** instead of **subspace convex-cyclic** operators.

## 2. Hahn-Banach Characterization

In this section, we start with our main theorems, and the following theorem shows us that extending our space under a few conditions remains Sub-Conv-Cyc operator to be the same, as a particular case with the same Sub-Conv-Cyc vector.

**Theorem 2.1**
Let $A \in \mathbf{B}(H)$ be a Sub-Conv-Cyc operator for $W$ if $V$ is an invariant subspace for $A$ ( $A(V) \subset V$ ) and $W \subset V$, then $A|_V : V \to V$ is a Sub-Conv-Cyc operator for W.

**Proof:**
Since we have $A$ is Sub-Conv-Cyc operator for $W$, then follow Definition 1.1 there exists an element like $x \in W$ such that

$$\overline{Orb(A,x) \cap W} = W. \qquad (1.1)$$

Besides, $N$ is an invariant subspace $A(V) \subset V$, then we will define

$$A|_V \,(a) = a. \qquad (1.2)$$

For being more specific, we can say $a \in W$ as we have $W \subset V$. So, we write equation (1.2) as

$$A|_V \,(a) = a, \quad for\; all \quad a \in W. \qquad (1.3)$$

Now, combine equations (1.1) and (1.3) we get

$$\overline{Orb(A|_V \,, x) \cap W} = W.$$

As a consequence, we will have $A|_V$, which is a Sub-Conv-Cyc operator over W.

**Definition 2.2** [2]
Let $W$ and V be subspaces of $X$. If $W \bigcap V = \{0\}$ and $W + V$ = X, and we say that W and V are integrating each other.

**Definition 2.3**
Let W and V be two subspaces of $H$ as they defined in Definition 2.2. We recall projection from W onto V when the function $J : H \to H$ defined as $J(x+y)=x$, where $x \in W$ and $y \in V$.

Similarly, we get the same results as in [10]:

**Lemma 2.4**
Given an operator $A \in \mathbf{B}(H)$, $V \subset H$ and $J$ the projection from $H$ onto $V$, then $V$ is an invariant subspace of $J$ if and only if $JA = JAJ$.

**Lemma 2.5**
Given an operator $A \in \mathbf{B}(H)$, and $J$ the projection from W onto $V$. If $JA = JAJ$, then $(JA)^k = JA^k$ for all $k \in \mathbf{N}$ where $\mathbf{N}$ is the set of non-negative integer numbers.

**Proof**

By using mathematical inductions, for $k$=1, it is clear. Let $(JA)^k = JA^k$ be true for $k=m$. Now for $k=m+1$, $(JA)^{m+1} = (JA)(JA)^m = JAJA^m = JAA^m = JA^{m+1}$.

Let us go out of the subspace that we work on it, with restricting our operator, as we see in the following Sub-Conv-Cyc operator on the small subspace $G$, and using projection like $J$ restricted over $W$ gives us an excellent result on $G$ as a Sub-Conv-Cyc operator.

**Theorem 2.6**

Let $W$ and $V$ be two integrate subspaces as defined in Definition 2.2 over $H$. And let $J$ be the projection as defined in Definition 2.3. Suppose that $AV \subset V$ (is invariant). If $A$ is a Sub-Conv-Cyc operator for some, $G \subset W$, then also $JA_{|W}$.

**Proof**

Since A is a Sub-Conv-Cyc operator for $G$, so we are sure that we have a Sub-Conv-Cyc vector $x \in G$ in such way $Orb(A,x) \cap G$ is dense G and as we know $G \subset W$, this directed as $J(Orb(A,x)) \cap G$ is comfortably dense in G hence

$$J\{x, Ax, A^{\acute{e}}x, \cdots\} \cap G = \{Jx, JAx, JA^{\acute{e}}x, \cdots\} \cap G = \{y, Ay, A^{\acute{e}}y, \cdots\} \cap G$$

where $Jx = y$ for all $x, y \in G$ thus, it is dense within G.

Also, since $V$ is an invariant by the given so, by Lemma 2.4 we have $JA= JAJ,$ and even by Lemma 2.5 we have $(JA)^k = JA^k$ , $\forall\, k \in N$ whenever

$$\begin{aligned} J(Orb(A,x)) &= J\{\alpha_0 + \alpha_1 Ax + \alpha_2 A^2 x + \cdots\} \\ &= \{J\alpha_0 + J(\alpha_1 Ax) + J(\alpha_2 A^2 x) + \cdots\} \\ &= \{\alpha_0 + \alpha_1 JAx + \alpha_2 JA\ \ x + \cdots\} \end{aligned}$$

On the other hand, for all $x \in G \subset W$,

$$\begin{aligned} Orb(JA_W, x) &= \{\alpha_0 + \alpha_1 (JA)x + \alpha_2 (JA)^2 x + \cdots\} \\ &= \{\alpha_0 + \alpha_1 JAx + \alpha_2 JA^2 x + \cdots\} \end{aligned}$$

Thus $J(Orb(A,x)) = Orb(JA_W, x)$, we get that $Orb(JA_W, x) \cap G$ shown dense set in G.

When we talk about Banach space X with an infinite dimension and separable, a Sub-Conv-Cyc operator distributes some of the properties that are the same as a W-hypercyclic operator or some time with a convex-cyclic operator. For an instant, it follows from the Definitions 1.1 that all Sub-Conv-Cyc vector for an operator $A$ on $X$ is directly a convex-cyclic vector for A.

We will prove oneself results as in [11], we will do it for a Sub-Conv-Cyc operator.

**Theorem 2.7**

If A is a bounded Sub-Conv-Cyc operator over a separable Banach space like X, then $\|A\| > 1$.

**Proof**

Suppose if possible that $\|A\| \leq 1$. Since $\|A\| = \sup\{\frac{\|Ax\|}{\|x\|}, \| x \| \neq 0\}$, with our operator because it's the generalized and sums of operators we choose it as $\|A\| = \sup\{\| A^n(x) \|, n \geq 1\}$, that is $\|A\| < \infty$, for all nonzero element $x \in W$, since A is a Sub-Conv-Cyc operator, then $\sup\{\|A^n x\|\} < \infty$, for each $n \geq 1$ and

$x \in W$ which implies that *Orb(A,x)* is bounded and countable that is, it is not dense in *W* which contradicts our assumption, so $\|A\| > 1$.

**Theorem 2.8**

Let *X* be Banach space with an infinite dimension and separable, and $W \subseteq X$ let *g* be a continuous linear functional such that *g,* not zero. Then *A* is a Sub-Conv-Cyc operator if and only if $\sup[g(Orb(A,x))] \to \infty$.

**Proof**

**"**If" part is clear since $\sup[g(Orb(A,x))] \to \infty$. means $\sup[g(Orb(A,x))]$ is not bounded, we can get *A* is a Sub-Conv-Cyc operator by using Theorem 2.7.

"Only if part," assume that $\sup[g(Orb(A,x))] \to \infty$. This implies that $\sup[\mathrm{Re}(g(Orb(A,x)))] \to \infty$. Suppose that *A* is not a Sub-Conv-Cyc operator, so we should have a polynomial say *q(x)* such that $q(x) \in W$ and $q(x) \in Orb(A,x)$, but $q(x) \in \overline{Orb(A,x) \cap W}$, then by using Hahn-Banach Theorem [5] there exists a continuous linear functional on *W* say *f* such that $\mathrm{Re}[g(x)] < \mathrm{Re}[g(q(x))]$ for all $Orb(A,x) \cap W$, we get that $\mathrm{Re}(g(Orb(A,x) \cap W))$is bounded above thus $\sup[g(Orb(A,x))] < \infty$. Which contradicts our given.

**Theorem 2.9**

Let *X* be Banach space with an infinite dimension and separable, and $W \subseteq X$. If *A* is Sub-Conv-Cyc operator, then $\sup\|A^* g^*\| \to \infty$ for each $0 \neq g^* \in X^*$.

**Proof**

We prove it by getting a contradiction by using Theorem 2.8 that there exists a linear functional $g^* \neq 0$ in $X^*$ such that $\|A^* g^*\| \leq \alpha$ for some positive scalar $\alpha$. Let *x* be any arbitrary vector in *X*. Then

$$|g^*(Ax)| = |(A^* g^*)x| \leq \|A^* g^*\|\|x\| \leq \alpha\|x\|,$$

consequently, we get

$$|g^* Orb(A,x)| \leq \alpha\|x\|.$$

Thus we get that *Orb(A,x)* is not dense in *W,* then *x* cannot be a Sub-Conv-Cyc vector for *A;* the selection of *x* gives us a contradiction for *A* been a Sub-Conv-Cyc operator.

## 3. Spectral Properties

This section is about the properties of the spectrum of *A,* where *A* is a Sub-Conv-Cyc operator.

**Theorem 3.1**

Let *X* be Banach space with an infinite dimension and separable, and $W \subseteq X$. If *A* is Sub-Conv-Cyc operator, then $\sigma(A) \cap \mathbf{T} \neq \phi$.

Before we prove Theorem 3.1, we need the following lemmas, and one can find their proofs in [3].

**Lemma 3.2**

Let $A \in \mathbf{B}(X)$ be an operator

1. Suppose that $\sigma(A) \subset \mathbf{D}$. Then there exists $a<1$ and $N \in \mathbf{N}$ such that $\|A^n(x)\| \leq a^n \|x\|$ for any $x \in X$ and all $n \geq N$.
2. Suppose that $\sigma(A) \subset \mathbf{C} \setminus \overline{\mathbf{D}}$. Then there exists a>1 and $N \in \mathbf{N}$ such that $\|A^n(x)\| \geq a^n \|x\|$ for any $x \in X$ and all $n \geq N$.

**Lemma 3.3**
Let *U* be a compact subset of *F*, and let *F* be a connected component of *U*. Assume that *F* is contained in some open set $\Psi \subset F$ Then one can find there exists a clopen (i.e., closed and open) subset $\Sigma$ of *U* such that $F \subset \Sigma \subset \Psi$..

**Proof** of Theorem 3.1
Suppose that $\sigma(A) \cap \mathbf{T} = \phi$. Then there exists $\sigma_1$ and $\sigma_2$ such that $\sigma = \sigma_1 \cup \sigma_2$ by Lemma 3.3, such that $\sigma_1 \subseteq \mathbf{D}$ and $\sigma_2 \subseteq \overline{\mathbf{D}}^c$. By Riesz-decomposition theorem2.10 [13] there exist integrated invariant subspaces $W_1$ and $W_2$ such that

$$\sigma(A|_{W_1}) \subseteq \sigma_1 \text{ and } \sigma(A|_{W_2}) \subseteq \sigma_2 .$$

Let $x \in X$. Then there exist $w_1 \in W_1$ and $w_2 \in W_2$ such that $x= w_1+w_2$, as $A|_W = A|_{W_1} \oplus A|_{W_2}$ , in case that $w_2 = 0$ we have $\|A^n x\| = \|(A|w_1)^n\| = \|(A|_{W_1})^n w_1\|$ and by Lemma 3.2 we get that it converges to zero. Which means $Orb(A,x) \cap W$ is bounded and consequently with any subspace $W_1$ or any others does not give us a dense in that subspace as clarified in [6], which means *A* is not a Sub-Conv-Cyc operator and finally, we get a contradiction.
In case if $w_2 \neq 0$, then

$$\|A^n x\| = \|A^n w_1 + A^n w_2\| \geq \|A^n w_2\| - \|A^n w_1\|$$

again by using Lemma 3.2, we get that $\|A^n w_2\|$ goes to infinity and $\|A^n w_1\|$ zero it follows that $\|A^n x\|$ goes to infinity, which tells us many elements they could be finite of $Orb(A,x) \cap W$ intersecting with $W_1$ and hence cannot be dense [6] in that subspace, and again we get a contradiction.

We could let $\sigma_1$ and $\sigma_2$ be empty as you can also enable $w_1 = 0$, and you get the same result.

**Remark 3.4**
As mentioned in [10] for an *M*- Hypercyclic operator and different with the convex-cyclic operator in [11] that all parts of the spectrum $\sigma_i(A)$ shouldn't meet **T** for any aspect subspace, also whenever *A* is a Sub-Conv-Cyc operator, we get the same result, and we show it by the same counterexample as follow:

**Example 3.5**
Let the operator T=$2B \oplus 3I$ , where *B* is the backward shift, and *I* is the identity operator, defined as to $T : \ell^2 \oplus \ell^2 \to \ell^2 \oplus \ell^2$ be a Sub-Conv-Cyc operator to the subspace $W = \{0\} \oplus \ell^2$ with a Sub-Conv-Cyc vector $(0,l)$, when *l* is vector Sub-Conv-Cyc for *B* we recognize that $\sigma(T) = 2\mathbf{D} \cup \{3\}$. Thus the same component does not intersect **T**.

**Theorem 3.6**
Let *X* be Banach space with an infinite dimension and separable. If *A* is Sub-Conv-Cyc operator, then *A* has a dense range.

**Proof**
Assume that *A* is a Sub-Conv-Cyc operator and *x* be a Sub-Conv-Cyc vector for it, we will try to show it by indirect method, by supposing that *A* has no dense range and let *R*(*A*) denotes the range of *A*. Then one can find a continuous linear functional *g* such that $g[R(A)] = \{0\}$. Then by Theorem 2.8 we should have that $\sup[\operatorname{Re}(g(Orb(A,x)))] \to \infty$. while $A^n x \in R(A)$ , $\forall n \geq 1$ it follows that $g(A^n x) = 0$ , $\forall n \geq 1$ . So,

$\sup[\mathrm{Re}(g(Orb(A,x)))] = \sup[\mathrm{Re}(\{A^n x, 0\})]$. It comes after by Theorem 2.8 that $x$ is impossible to be a Sub-Conv-Cyc vector, which is an opposite to our assumption; thus, $A$ must get a dense range [6].

**Theorem 3.7**

If $A$ is Sub-Conv-Cyc operator on W. Then $Ker(A^* - \lambda) \subseteq W^{\perp}$ for all $\lambda \in \mathbf{C}$.

**Proof**

Let $\ker(A^* - \lambda) \not\subseteq W^{\perp}$ for all non-zero $\lambda \in \mathbb{C}$, we want to show that $T$ is not a subspace convex cyclic operator.

Since $\ker(A^* - \lambda) \not\subseteq W^{\perp}$ then there exist $z \in \ker(A^* - \lambda)$ such that $z \notin W^{\perp}$, so

$z \in \ker(A^* - \lambda)$ and $z \notin W^{\perp}$,

$(A^* - \lambda)z = 0$ and $< z, w > \neq 0$, $for\ all\ w \in W\ and\ z \in X$,

$A^* z = \lambda z$ as a consequence we get that,

$A^{n^*} z = \lambda^n z\ for\ all\ n \in \mathbb{N}$.

Now for each $n \in \mathbb{N}$, $< \lambda^n z, w > = < A^{n^*} z, w > = < z, A^n w >$ we have two cases.

Case I: $< z, A^n z > = 0\ for\ all\ n \in \mathbb{N}$.

Then $z = 0$ {which is contradicted to our assumption that said $< z, w > \neq 0$ $for\ all\ w \in W\ and\ any\ arbitrary\ z \in X.$}

Or $A^n w = 0$ {which is contradicted to our assumption that said for $n = 1, < z, \lambda w > = \bar{\lambda} < z, w >= 0$}.

Case II: $< z, A^n w > \neq 0\ for\ all\ n \in \mathbb{N}$.

We recognize that $A$ is no more Sub-Conv-Cyc over $W$, because we never reach 0 vectors. And since $M$ is a subspace, it should contain 0, so we get a contradiction.

**Definition 3.8**

Let $A \in \mathbf{B}(H)$ be a Sub-Conv-Cyc operator for W. We denote the orbits collection of $W$ by $Orb(A,W)$ under such Sub-Conv-Cyc operator A, and we denote $W = \{w_1, w_2, w_3, \cdots\}$, then

$$Orb(A,W) = \{\alpha_0 w_1, \alpha_1 Aw_1 + \alpha_2 Aw_1 + \alpha_3 Aw_1 + \cdots\} \cup \{\alpha_0 w_2, \alpha_1 Aw_2 + \alpha_2 Aw_2 + \alpha_3 Aw_2 + \cdots\} \cup \cdots$$

**Theorem 3.9**

Let $A \in \mathbf{B}(H)$ be a Sub-Conv-Cyc operator for W, of $W \subset H$. Then for any number like $\lambda$

$$Span[(\lambda I - A)Orb(A,W)]^{\perp} \subseteq Span[Orb(A,W)]^{\perp}.$$

**Proof**

Suppose $w \in W$ is a Sub-Conv-Cyc vector for $A$ and $g$ is a linear functional and not equal to zero, which is orthogonal to some set like this subset $(\lambda I - A)Orb(A,W)$, since the set $Orb(A,W)$ is an invariant subspace because $A$ is a Sub-Conv-Cyc operator then $A(Orb(A,W)) \subseteq Orb(A,W)$, hence

$$g(A^n w) = \lambda^n g(w) \tag{3.1}$$

for all $w \in Orb(A,W)$ and $n \in \mathbf{N}$.

If $g\big|_{Orb(A,W)} \neq 0$, using Equation (3.1) and after expanding Definition 3.8 we get that $Orb(A,W) = \{\alpha_0 w_1, \alpha_1 Aw_1 + \alpha_2 Aw_1 + \alpha_3 Aw_1 + \cdots\} \bigcup \{\alpha_0 w_2, \alpha_1 Aw_2 + \alpha_2 Aw_2 + \alpha_3 Aw_2 + \cdots\} \bigcup \cdots$
Since $W$ is a subspace, then $W \subseteq Orb(A,W)$, it is clear that $g\big|_W \neq 0$. Thus $g(W)$ is equivalent to **R** or **C**, and we see that $g[Orb(A,W)]$ is also will be **R** or **C**. From the other side $x$ is a convex-cyclic vector for $A$. Then by definition $\overline{Orb(A,w)} \bigcap W = W$ that is, $W \subseteq \overline{W}$ and $W \subseteq \overline{Orb(A,w)}$. Hence

$$Orb(A,W) \subseteq \overline{Orb(A,w)},\ \text{since it is dense in } W.$$

By taking $g$ to both sides, we get

$$g[Orb(A,W)] \subseteq g\left[\overline{Orb(A,w)}\right]$$
$$g[Orb(A,W)] \subseteq g\left[A^n(w)\right]$$
$$\overline{g[Orb(A,W)]} \subseteq \overline{g\left[A^n(w)\right]}$$

By the above fact and using Equation (3.1), we get $\mathbf{R} \subseteq \overline{\{\lambda^n(w), \forall \quad n \in \mathbf{N}\}}$ since $\overline{\{\lambda^n(w), \forall \quad n \in \mathbf{N}\}} \neq \mathbf{R}$ where $\lambda$ a fixed scalar, so we get a contradiction. Hence, $g$=0 which means that

$$Span[(\lambda I - A)Orb(A,W)]^{\perp} \subseteq Span[Orb(A,W)]^{\perp}.$$

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